Systems & Control: Foundations & Applications

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This book is dedicated to:

Ann, Ingmar, Ula, Dmitri, and Mirabella (Ali Saberi)

My parents (Anton A. Stoorvogel)

Jayanth (Pedda Sannuti)
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Preface

As soon as we (AS and PS) completed writing the book on $H_2$ Optimal Control, another task of equal magnitude was laid to our charge. This task was to work on filtering and related topics. This book releases us from this charge. In this endeavor, we are fortunate to have found a capable person in our friend and colleague (AAS) who helped us release our burden.

The subject of filtering is indeed vast and immense, much more so than the subject of $H_2$ Optimal Control. In this work, we have tried to present what we believe to be the fundamental issues of filtering. The book is not intended to give a chronological development of filtering from a historical point of view. A vast number of books already do so. Our intent here is to develop from our perspective the complete theory of filtering and various design methodologies associated with it along with their practical implementations. In this respect, we present here a state-of-the-art view of exact and almost input-decoupled filtering, $H_2$, and $H_{\infty}$ filtering and inverse filtering issues, and include an application of filtering and inverse filtering to fault detection, isolation, and estimation. Most of the work reported here arose out of the research conducted by one or more of us and sometimes in collaboration with our students and colleagues.

Supposedly, young F. Scott Fitzgerald proclaimed in 1920 right after his first novel (This Side of Paradise) that, “an author ought to write for the youth of his own generation, the critics of the next, and the school masters of ever afterward”. It is very presumptuous of us to say the same. Nevertheless, we strived to do so. Also, it is said that one must do his/her work and renounce the fruits of it to the ONE and MANY that pervade the universe. We have done our work. Let its fruits be fruits of many.

Our intended audience includes practicing engineers, graduate students, and researchers in filtering, signal processing, and control. An appropriate background for this book is a first graduate course in state-space methods as well as a first graduate course in filtering.

No work of this magnitude and nature can be undertaken without many sacrifices. Our families are the ones who sacrificed the time we could have spent with them otherwise. Needless to say that we owe a debt of gratitude to our families, and it is natural that we dedicate this book to them. We certainly also owe a debt of gratitude to our editor, Dr. Tamer Başar, and the editorial staff at Birkhäuser. Our special thanks go to the copy editor for a meticulous editing that improved the text.

Ali spent countless number of hours brooding over the manuscript of this book at Bucer’s, the great coffee house of Moscow, Idaho. The melodious atmosphere at Bucer’s nurses many bruised souls to their vitality. Ali acknowledges the con-
The publication of this book marks over two decades of collaboration between AS and PS. Each would like to acknowledge this long partnership and friendship and express their hopes that this will continue undiminished for many more years.

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1

Introduction

1.1 Introduction

*Estimation theory* and, in the same breath, *filtering theory* is vast and rich in the literature and is central to a wide variety of disciplines, including control, communications, and signal processing. Also, it is relevant to such diverse areas as statistics, economics, bioengineering, and operations research. The terms *estimation* and *filtering* evoke many and varied responses among engineers and scientists. In its primary level, “estimation” is the process of arriving at a value for a desired and unknown variable from certain observations or measurements of other variables related to the desired one but contaminated with *noise*. Although one could trace the origins of estimation back to ancient times, Karl Friederich Gauss is generally acknowledged to be the forefather of what is now referred to as estimation theory. In his quest to predict the motions of planets and comets from telescopic measurements, Gauss at the age of 18 formulated the now well-known method of *least squares*. In modern times and in particular in the second half of the twentieth century, filtering theory has become synonymous with estimation theory mainly in the engineering literature. It might look odd that the term “*filter*” would apply to an “*estimator*”. In its common use, a “filter” is a physical device that can separate the wanted and unwanted fractions of a mixture. In electronics, a filter is seen as a circuit with a frequency-selective behavior, and thus can attenuate certain undesired components of the input signal and pass to the output certain desired components of the input signal. This notion of separating a signal into certain desired and undesired components can be stretched further in *signal processing*, where a signal contaminated with noise is enhanced or reconstructed by eliminating noise as much as possible. In the 1930s and 1940s, such a notion of filtering was stretched even further to separate the desired *signals* from *noise*, both of which were characterized by their covariance functions. The pioneering work of Kolmogorov and Wiener used a statistical characterization of the probability distributions of signals and noise in forming an optimal estimate of the signal, given the sum of the signal and noise. In the early 1960s, Kalman changed the then conventional formulation of the problem, in which the covariance of the signal process is given, to a new formulation in which a *model* for the signal process is given and views the signal as the output of a linear finite-dimensional dynamical
system driven by a white noise process. Thus, Kalman championed a filter (which is now affectionately called the Kalman filter) to solve what is now known as the linear quadratic Gaussian estimation problem. Ever since the seminal work of Kalman, the concept of filtering has broadened in several ways, and the research in the field of filtering and estimation has been pursued on a variety of fronts. Indeed, with the advent of Kalman filtering, the word filtering assumed a meaning that is well beyond the original idea of separating the components of a mixture. Moreover, it has come to include the solution of an inversion problem in which one knows how to represent the measurable variables as functions of the variables of principle interest, then inverts this functional relationship, and in doing so estimates the independent variables as inverted functions of the measured variables. Such an inverse filtering problem occurs in many practical applications where a signal of interest passes through a system whose measured output is corrupted by noise. When the system under consideration is a linear system, the measured output signal, although corrupted by noise, is the convolution of the desired input signal with the impulse response of the system, which is either partially or fully known. In this context, inverse filtering is also known as deconvolution, and it plays an important role, especially in the areas of signal processing and communication.

Thus, the word filter is viewed now as having a variety of functions, some of which can be summarized as follows:

- A filter estimates instantaneously or with a specified delay an unknown signal, normally a function of state and input of a system using the measured outputs.
- A filter separates a signal into its component signals and, in particular, separates the desired signal from the noise.
- A filter solves an inversion problem in which one seeks to estimate the intended inputs of a system from the measured outputs of it.
- A filter is used as a learning tool by which a certain signal is tracked, estimated, and predicted.

Filters and, in particular, Kalman filters are now used ubiquitously in several diverse areas. The principal uses of Kalman filtering have been in modern control systems, in the tracking and navigation of all sorts of vehicles, and in the prediction of future behavior. The list of Kalman filter applications is endless and includes not only satellite navigation, trajectory estimation, guidance, video and laser tracking systems, and radars, but also oil drilling, water and air quality control, geodetic surveys, and many others. Indeed it is not an overstatement to assert that Kalman filter with its many extended formats represents one of the most widely and demonstrably useful results that emerged in the twentieth century. It has enabled humankind to do many things that could not have been done otherwise, and it has become as indispensable as silicon in the makeup of electronic systems.
1.2 Filtering problems

To give a precise meaning to the word filtering, consider the block diagram of a typical setup as depicted in Figure 1.1. In Figure 1.1, a given system is excited by two kinds of input signals: an intended excitation signal and noise. The noise signals arise from a variety of causes, such as unknown external input signals, plant noise, model uncertainties of the given system, as well as measurement or observation noise. The job of the filter is to estimate the desired output of the given system by using the measured output of the system. Obviously, to design a filter that does such an estimation, one has to have an associated performance index that is a function of the estimation error, which is the difference between the desired and the estimated output. The job of filter design is to render the chosen performance index as small as possible. The performance index or measure can be defined in several ways. One remarkably common performance index is the integral square of the estimation error in which small values of the error are weighed relatively less than large values of the error. Such a problem has been labeled as the least-squares estimation problem, least (minimum) mean square estimation, or filtering problem. A filter that can achieve the minimum of the chosen performance index is called an optimal filter. Instead of seeking an optimal filter, typically one could also seek to render the performance index less than a prescribed value.

![Figure 1.1: Block diagram of a typical setup](image)

In estimation problems, one encounters three different divisions:

- Depending on whether we need to estimate the future, the current, or the past of the signals, we deal with a prediction, filtering, or smoothing (delayed filtering) problem. Nevertheless, independent of whether it is a prediction, filtering, or smoothing (delayed filtering) problem, one often calls it a filtering problem.

- In another type of classification, depending on whether observations are made over a finite or a semi-infinite interval of time, one deals with a filtering problem having a finite or infinite horizon of observations that are, respectively, referred to as finite or infinite horizon filtering problems.
• One could also divide the filtering problems encountered in the literature into four categories depending on the assumptions made on the noise characteristics.

(i) In the first category, certain statistical assumptions (typically knowledge of covariance functions and power spectral densities) are made on the noise process.

(ii) In the second category, no statistical assumptions are made, and the noise is essentially considered as unknown except that it has a finite root-mean-square (RMS) value.

(iii) In the third category, noise signals can be divided into two subsets. As in the first category, certain statistical assumptions are made on the noise signals belonging to one set, while as in the second category, no assumptions are made on the noise signals belonging to the other set.

(iv) For each of the above three categories, one can create another subcategory by adding other unknown input signals which contain a linear combination of sinusoidal signals, each of which has an unknown amplitude and phase but known frequency.

This book studies various classes of filtering problems and the corresponding filters that solve them. Various classes of filtering problems are defined based on different assumptions made on the noise characteristics and on different performance measures (indexes) that are adapted. Each class of filtering problem is studied in depth regarding (1) the existence of a filter that solves it, (2) uniqueness of such a filter, and (3) designing such a filter or filters while showing the flexibility in assigning poles of the filter or filters.

The first filtering problem arises by seeking that the RMS value of the estimation error be zero (i.e., by seeking that the estimation error tend asymptotically to zero as time progresses to infinity) irrespective of the nature of noise or input to the given system. Such a problem demands exact estimation, and it dictates that the transfer matrix from the inputs to the estimation error be identically zero. In other words, in such a problem, we seek a filter that estimates the desired output in such a way that the error in the estimation of the desired output is completely decoupled from the input(s). For this reason, we can call the problem we pose here the exact input-decoupling filtering problem, or, for short, the EID filtering problem, and the filters that solve such a problem the exact input-decoupling filters or EID filters. EID filtering is studied in depth in Chapter 7.

The EID filtering problem stated above demands a severe performance measure, namely that the transfer matrix from the input to the estimation error be identically zero. As such, the EID filtering problem is not always solvable. It is natural then to think of methods of relaxing the performance requirements so that the solvability conditions can possibly be weakened and thus allow us to deal with a larger class of systems. There are several ways by which the performance requirements can be weakened. We plan to relax the performance requirements progressively layer by layer to form a hierarchy of problems as outlined below.
We can introduce the first layer of relaxing the performance requirements by seeking that the RMS value of the estimation error be “almost zero”, or, equivalently, “arbitrarily small” or “as small as desired”, instead of being identically zero. To be more precise, we try to find a family of filters parameterized by some positive $\varepsilon$ such that when applied to the system, the RMS value of the estimation error converges to zero as $\varepsilon \downarrow 0$. Such a problem can be termed as an almost-input-decoupled (AID) filtering problem or, for short, an AID filtering problem. The filters that solve the AID filtering problem can be termed not surprisingly as AID filters.

In AID filtering, we primarily focus on two cases depending on the assumptions made on the noise characteristics, whether we have known noise statistics or unknown noise statistics. Clearly, the noise can be modeled as a stochastic process. Whenever the noise is a stochastic process, all signals, i.e., the state, the measured output, and the desired output of the given plant or system, are naturally stochastic processes. In one framework, the noisy input to the given system can be modeled as a zero mean wide sense stationary white noise stochastic process of power spectral density (PSD) equal to an identity matrix, which can be called simply as white noise of unit intensity. In fact, one can assume without much loss of generality the input as any zero mean wide sense stationary stochastic process of known PSD, which is not necessarily a white noise. In this regard, one can recall easily the well-known fact that one can always generate a wide sense stationary stochastic process as the output of a linear time-invariant system driven by white noise of unit intensity so that the PSD of such a generated stochastic process approximates arbitrarily closely any known PSD of a wide sense stationary stochastic process. The needed linear time-invariant system to do so can always be appended to the given system. As such, there is not much loss of generality in assuming the input to be zero mean white noise of unit intensity.

Thus, in our study of AID filtering, as a first case we can model the input as a zero mean wide sense stationary white noise of unit PSD. Then, we seek a family of parameterized filters such that when applied to the system, the RMS norm of the estimation error signal converges to zero as the parameter tends to zero. By the definition of the $H_2$ norm of a transfer matrix, this is equivalent to demanding that the $H_2$ norm of the transfer function from the noise input to the estimation error be arbitrarily small. As such, such an AID filtering problem is referred to as the $H_2$ AID filtering problem, and the family of filters that solves such a problem as the family of $H_2$ AID filters. $H_2$ AID filtering is studied in depth in Chapter 8.

Unlike the first case discussed, we can assume no statistical information on the input except that it has a finite RMS norm. Under such an unknown input, we seek to render the ratio of the RMS value of the error to the RMS value of the input arbitrarily small. By the definition of the $H_\infty$ norm of a transfer matrix, this is equivalent to demanding that the $H_\infty$ norm of the transfer function from the noise input to the estimation error be arbitrarily small. As such, such an AID filtering problem is referred to as the $H_\infty$ AID filtering problem, and the family of filters that solves such a problem as the family of $H_\infty$ AID filters. $H_\infty$ AID filtering is studied in depth in Chapter 9.
6 1. Introduction

As discussed, AID filtering seeks to have the RMS norm of the estimation error signal as small as desired. We can relax such a performance requirement even further by seeking that the RMS norm of the error signal be as small as possible rather than as small as desired. This requirement leads us to optimally input-decoupling (OID) filtering. As in AID filtering, we can divide the OID filtering problems into two categories depending on the assumptions made on the noise characteristics.

At first, we can follow the direction set by AID filtering under white noise input. That is, we can assume that the input to the given system is a white noise of unit intensity and seek to make the RMS norm of the error signal as small as possible. Once again, by the definition of the $H_2$ norm of a transfer matrix, this is equivalent to demanding that the $H_2$ norm of the transfer function from the noise input to the estimation error be minimized. As such, such an OID filtering problem under white noise input is referred to as an $H_2$ OID filtering problem, and the filters that solve such a problem as the $H_2$ OID filters. $H_2$ OID filtering is studied in depth in Chapter 10.

What we called an $H_2$ OID filtering problem is indeed the celebrated Kalman filtering problem. Most available literature on Kalman filtering deals only with what are known as regular filtering problems, which rely on a crucial assumption that all measured observations of the given system are fully corrupted by the white Gaussian noise process. On the other hand, many practical applications exist where a, what can be called singular, situation arises. Such cases arise, for instance, when observations are corrupted by colored noise or when some or all observations are being modeled as noise free. Also, the formalism of the Wiener filtering problem when transformed to a Kalman filtering problem leads to a singular situation. This book considers general singular filtering problems.

Another concept highly tied to $H_2$ OID filtering is the suboptimally input-decoupled (SOID) filtering problem. In the absence of a formal definition of suboptimality, any filter that is not optimal can be construed as a suboptimal filter. A good definition of suboptimality can be given through the notion of attaining an RMS norm of the error signal arbitrarily close to its infimum. In this regard, a sequence or a family of filters can be called suboptimal filters if one can select a filter from the family such that the resulting RMS norm of the error signal is within an arbitrarily specified value from its infimum. The filtering problem that arises in this regard can be termed as an $H_2$ SOID filtering problem, and the family of filters that solves such a problem as the family of $H_2$ SOID filters. $H_2$ SOID filtering is studied in depth in Chapter 10.

In the $H_2$ OID and $H_2$ SOID filtering considered above, we assume that the input to the given system is a white noise of unit intensity. We can follow another direction in which no statistical information about the input is available except that it has a finite RMS value. In this connection, for a given filter, we can express the performance by the smallest number $\gamma$ for which the RMS norm of the error signal for any input is always less than or equal to $\gamma$ times the RMS norm of the input. Then, one can pose a filtering problem as the problem of finding a filter that can achieve the smallest possible value for $\gamma$, which can be denoted by $\gamma_{sp}^*$ or $\gamma_p^*$ depending on whether we consider the class of strictly proper or
1.2 Filtering problems

proper filters. Clearly then, an optimal filter achieves the minimum possible performance, namely, $\gamma_{sp}^*$ or $\gamma_p^*$. In the case of a suboptimal filter, for a specified number $\gamma > \gamma_{sp}^*$ or $\gamma > \gamma_p^*$, one seeks a filter that achieves the RMS norm of the error signal for any input less than or equal to $\gamma$ times the RMS norm of the input. Such a suboptimal filter can be termed a $\gamma$-level suboptimal filter. One can, of course, seek in general to design an optimal filter or a $\gamma$-level suboptimal filter.

However, essentially most available literature bypasses seeking optimal filters and focuses only on $\gamma$-level suboptimal filters. A primary reason to do so is that for general systems no elegant analytic formula exists that enables one to compute the infimum performance measure $\gamma_{sp}^*$ or $\gamma_p^*$. Only numerical approximations of computing $\gamma_{sp}^*$ or $\gamma_p^*$ exist, and this obviously implies that seeking $\gamma$-level suboptimal filters is very natural. Also, there is a secondary reason to seek such suboptimal filters, namely, that the existence conditions for an optimal filter are prohibitively complex. Moreover, for engineering applications, often suboptimal filters suffice.

By the definition of the $H_\infty$ norm of the transfer function of a system, the $\gamma$-level suboptimal filtering problem we posed above is equivalent to demanding that the $H_\infty$ norm of the transfer function from the noise input to the estimation error be less than $\gamma$. As such, such a problem is referred to as a $\gamma$-level $H_\infty$ suboptimally input-decoupling filtering problem or, for short, as a $\gamma$-level $H_\infty$ SOID filtering problem. The filter that solves such a problem can obviously be called the $\gamma$-level $H_\infty$ SOID filter. These are studied in depth in Chapter 11.

All the filtering problems introduced above are reconsidered with additional unknown input signals containing a linear combination of sinusoidal signals, each of which has an unknown amplitude and phase but known frequency. We have used the qualifier “generalized” to refer to each of the above problems when such additional inputs are present. For instance, the $H_2$ OID, $H_2$ SOID, and $\gamma$-level $H_\infty$ SOID filtering problems described above when additional sinusoidal inputs of unknown amplitude and phase but known frequency are present are, respectively, referred to as generalized $H_2$ OID, generalized $H_2$ SOID, and generalized $\gamma$-level $H_\infty$ SOID filtering problems. The generalized $H_2$ OID and the generalized $H_2$ SOID filtering are studied in depth in Chapter 12, whereas the generalized $\gamma$-level $H_\infty$ SOID filtering is studied in depth in Chapter 13.

In the OID filtering problems we discussed above, we assume that either the input to the system is a white noise of unit intensity or no statistical information about it is available except that it has a finite RMS value. Obviously, one can have a mixed case in which a part of the input is modeled as a white noise and the other part as a signal for which no statistical information is available except that it has a finite RMS value. The literature contains some such mixed filtering work, including the one authored by us [77]. However, research on mixed filtering problems is still progressing and is not complete. At this time, the research is not mature enough to be dealt with in this book.

All the filtering problems enumerated above are discussed in detail in this book for both continuous- and discrete-time systems. In particular, the solvability condition for each problem is developed. Also, the conditions for the uniqueness of a filter that solves a given problem are given. Whenever a problem is solvable, ex-